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APPLICATIONS OF GROUP COHOMOLOGY TO SPACE CONSTRUCTIONS

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ABSTRACT. From a short exact sequence of crossed modules $1 \to K \to H \to \bar{H} \to 1$ and a 2-cocycle $(\phi, h) \in Z^2(G; H)$, a 4-term cohomology exact sequence

$$H^1_{ab}(G; Z) \to H^1_{(\bar{\phi},\bar{h})}(G; \bar{H}, \bar{Z}) \xrightarrow{\delta} \bigcup\{H^2_\psi(G; K): \psi_{\text{out}} = \phi_{\text{out}}\} \to H^2_{ab}(G; Z)$$

is obtained. Here the first and the last term are the ordinary (=abelian) cohomology groups, and Z is the center of the crossed module H. The second term is shown to be in one-to-one correspondence with certain geometric constructions, called Seifert fiber space construction. Therefore, it follows that, if both the end terms vanish, the geometric construction exists and is unique.

1. Definitions, notations and main results. For any group G and an element $a \in G$, the symbol $\mu(a)$ denotes the conjugation by a; that is, $\mu(a)(x) = axa^{-1}$ for every $x \in G$. The group of all automorphisms of G is denoted by $\operatorname{Aut}(G)$; $\operatorname{Inn}(G)$ is the group of all inner automorphisms $\mu(a)$.

A *crossed module* is a 4-tuple (H, ρ, Π, Φ) where H and Π are groups; $\rho\colon H \to \Pi$ and $\Phi\colon \Pi \to \operatorname{Aut}(H)$ are group homomorphisms satisfying

$$\Phi \circ \rho = \mu, \quad \mu(x) \circ \rho = \rho \circ \Phi(x)$$

for every $x \in \Pi$.

The kernel of ρ, called the center of the crossed module, is denoted by Z. It is a subgroup of $Z(H)$, the center of H.

Let G be a group. A *2-cocycle* of G with coefficient in a crossed module (H, ρ, Π, Φ) is a pair of maps (ϕ, h), where $\phi\colon G \to \Pi$ and $h\colon G \times G \to H$ are maps (not homomorphisms) satisfying

$$\phi(x)\phi(y) = \rho(h(x,y))\phi(xy),$$
$$\Phi(\phi(x))(h(y,z))h(x,yz) = h(x,y)h(xy,z)$$

for all $x, y, z \in G$. We always require h be normalized; that is,

$$h(x,1) = 1 = h(1,x)$$

for every $x \in G$.

The set of all such 2-cocycles of G with coefficients in a crossed module (H, ρ, Π, Φ) is denoted simply by $Z^2(G; H)$. For $(\phi, h) \in Z^2(G; H)$ and $b \in H$, the

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element $\Phi(\phi(x))(b)$ of H will be denoted by ${}^{\phi(x)}b$, or more simply by ${}^{x}b$, if there is no confusion as to which ϕ is being used.

Let $(\phi, h) \in Z^2(G; H)$. For any map $\lambda\colon G \to H$, we define $h_\lambda\colon G \times G \to H$ by

$$h_\lambda(x, y) = \lambda(x) \cdot {}^{x}\lambda(y) \cdot h(x, y) \cdot \lambda(xy)^{-1}$$

for every $x, y \in G$. Two cocycles $(\phi, h), (\psi, k) \in Z^2(G; H)$ are said to be *cohomologous* if there exists a map $\xi\colon G \to H$ for which

$$\psi = (\rho \circ \xi) \cdot \phi, \quad k = h_\xi.$$

More precisely, $k(x, y) = \xi(x) \cdot \Phi(\phi(x))(\xi(y)) \cdot h(x, y) \cdot \xi(xy)^{-1}$. This is an equivalence relation on the set $Z^2(G; H)$. We denote the set of cohomology classes of $Z^2(G; H)$ by $H^2(G; H)$, and call it the *second cohomology* of G with coefficient in the crossed module (H, ρ, Π, Φ).

With a fixed 2-cocycle $(\phi, h) \in Z^2(G; H)$, a 1-*cocycle* is a map $\lambda\colon G \to H$ satisfying $h_\lambda = 1$. That is,

$$\lambda(x) \cdot {}^{x}\lambda(y) \cdot h(x, y) \cdot \lambda(xy)^{-1} = 1.$$

The set of all 1-cocycles is denoted by $Z^1_{(\phi,h)}(G; H)$. Two 1-cocycles λ and η are *cohomologous* if there exists an element $c \in H$ satisfying

$$\eta(x) = c \cdot \lambda(x) \cdot {}^{x} c^{-1}$$

for all $x \in G$. The set of all cohomology classes of 1-cocycles is denoted by $H^1_{(\phi,h)}(G; H)$, and is called the *first cohology* of G with coefficient in the crossed module (H, ρ, Π, Φ) with respect to the 2-cocycle (ϕ, h).

A *morphism* between crossed modules $(H_1, \rho_1, \Pi_1, \Phi_1) \to (H_2, \rho_2, \Pi_2, \Phi_2)$ is a pair of homomorphisms

$$f\colon H_1 \to H_2, \quad \tau\colon \Pi_1 \to \Pi_2$$

satisfying

$$\rho_2 \circ f = \tau \circ \rho_1, \quad f \circ \Phi_1(x) = \Phi_2(\tau(x)) \circ f$$

for every $x \in \Pi_1$. Let $(H_i, \rho_i, \Pi_i, \Phi_i)$ $(i = 1, 2, 3)$ be crossed modules. A diagram

$$\begin{array}{ccccccccc} & & \Pi_1 & \xrightarrow{\sigma} & \Pi_2 & \xrightarrow{\tau} & \Pi_3 & & \\ & & \uparrow{\scriptstyle \rho_1} & & \uparrow{\scriptstyle \rho_2} & & \uparrow{\scriptstyle \rho_3} & & \\ 1 & \to & H_1 & \xrightarrow{i} & H_2 & \xrightarrow{j} & H_3 & \to & 1 \end{array}$$

is called a *short exact sequence* of crossed modules if the bottom sequence is a short exact sequence of groups, (i, σ) and (j, τ) are morphisms of crossed modules, σ is an isomorphism, and τ is surjective. In this paper, we shall assume that $\Pi_1 = \Pi_2$ and $\sigma = identity$, $\Pi_3 = \Pi/\rho_1(H_1)$ and τ is the natural quotient homomorphism.

Functoriality. Let $(f, \tau)\colon (H_1, \rho_1, \Pi_1, \Phi_1) \to (H_2, \rho_2, \Pi_2, \Phi_2)$ be a morphism. For any group G, there is an induced map

$$H^2(G; H_1) \to H^2(G; H_2)$$

sending $\langle \phi, h \rangle$ to $\langle \tau \circ \phi, f \circ h \rangle$. Also, for a fixed element $(\phi, h) \in Z^2(G; H_1)$ there is an induced map

$$Z^1_{(\phi,h)}(G; H_1) \to Z^1_{(\tau\circ\phi, f\circ h)}(G; H_2)$$

sending λ to $f \circ \lambda$.

The notation $H^2(G;H)$ denotes the cohomology with coefficient in a crossed module H. As we shall see later, even when H is abelian, $H^2(G;H)$ is different from the ordinary group cohomology. In fact, when $\Pi = \mathrm{Aut}(H)$, $H^2(G;H)$ is the union of all abelian group cohomology groups with all possible G-module structures on H. The abelian group cohomology will be denoted by $H^2_{ab}(G;H)$ with a specific G-module structure understood.

In §2, we shall develop a "long exact cohomology sequence" from a short exact sequence of crossed modules. The main result states that: Let $1 \to K \to H \to \bar{H} \to 1$ be a short exact sequence of crossed modules, and let $Z = \mathrm{Ker}\{\rho: H \to \Pi\}$. Then there exists a long exact sequence

$$H^1_{ab}(G;Z) \xrightarrow{\text{ev}} H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z}) \xrightarrow{\delta} H^2(G;K) \xrightarrow{i} H^2(G;H).$$

The first map is an 'evaluation map' coming from the action of $H^1_{ab}(G;Z)$ on the set $H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z})$.

In §3, we show that there is a simply transitive action of $H^2_{ab}(G;Z)$ on $H^2_{\phi}(G;H)$ which gives rise to a one-one correspondence between these sets. When Φ is injective, the above exact sequence becomes

$$\begin{aligned} H^1_{ab}(G;Z) \xrightarrow{\text{ev}} H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z}) &\xrightarrow{\delta} \bigcup\{H^2_{\psi}(G;K): \psi_{\text{out}} = \phi_{\text{out}}\} \\ &\xrightarrow{i} H^2_{\phi}(G;H) \cong H^2_{ab}(G;Z). \end{aligned}$$

In §4, we apply the previous results to group pairs. Let K be a normal subgroup of H. Then $1 \to K \to H \to \bar{H} \to 1$ becomes a short exact sequence of crossed modules. Let $1 \to H \to \mathcal{E} \to G \to 1$ be exact, K normal in $\mathcal{E}$ so that $\mathcal{E}$ represents a 2-cocycle $(\phi, h) \in Z^2(G;H)$. Then the second term $H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z})$ in the above exact sequence is shown to be in one-to-one correspondence with the set of Π-equivalence classes of group extensions E of K by G together with embeddings of E into $\mathcal{E}$. The main result of the first four sections is summarized in (4.3), which state the following:

Let K be a normal subgroup of H with AEP, and let $1 \to H \longrightarrow \mathcal{E} \longrightarrow G \to 1$ be an exact sequence of groups. If $H^2_{ab}(G;Z(H)) = 0$, then for any group extension $1 \to K \to E \to G \to 1$ whose abstract kernel inducing the abstract kernel for $\mathcal{E}$, there exists an injective homomorphism $\theta: E \to \mathcal{E}$ which extends the inclusion of K into H and induces the identity on G. If, furthermore, $H^1_{ab}(G;Z(H)) = 0$, then such a homomorphism θ is unique up to conjugation by elements of H.

In the last section, we prove a necessary condition for the existence and uniqueness of the Seifert fiber space construction for general Lie groups, using all the machinery that we developed earlier. For a explicit statement, see (5.2). As an application, we reprove the main theorem of [**KLR**] which states the existence and uniqueness of Seifert fibered manifolds with typical fiber a nilmanifold. For a motivation of such a construction of embeddings of E into $\mathcal{M}(W,L) \circ (\mathrm{Aut}(L) \times \mathcal{H}(W))$, see [**KLR** or **LR1**].

The algebra in the first four sections is not only important for the topological construction in §4, but also interesting in itself. A more powerful application of this algebraic result will be given in a forthcoming paper [**LR2**].

2. Long exact sequence. Let $(K, \rho, \Pi, \Phi), (H, \rho, \Pi, \Phi)$ and $(\bar{H}, \bar{\rho}, \bar{\Pi}, \bar{\Phi})$ be crossed modules and let

$$\begin{array}{ccccccccc} & & \Pi & \xrightarrow{\text{id}} & \Pi & \xrightarrow{\tau} & \bar{\Pi} & & \\ & & {\scriptstyle\rho}\uparrow & & \uparrow{\scriptstyle\rho} & & \uparrow{\scriptstyle\bar{\rho}} & & \\ 1 & \longrightarrow & K & \xrightarrow{i} & H & \xrightarrow{j} & \bar{H} & \longrightarrow & 1 \end{array}$$

be a short exact sequence of crossed modules. We try to get a "long exact sequence" of cohomology sets from this short exact sequence of coefficient modules. The theory developed here resembles those theories in [**D**, **LR**, and **I**]. However, the reader will find the concept of relative coefficients is new and the long exact sequence that we will get is very useful in the topological construction in §5. Furthermore, our long exact sequence can be calculated easily in many cases because both the end terms are ordinary abelian cohomologies.

NOTATION. $Z = \operatorname{Ker}\{\rho: H \to \Pi\}$, the center of the crossed module (H, ρ, Π, Φ).

Notice that Z is a subgroup of the center $Z(H)$ of the group H. This group will play a crucial role in formulating a long exact sequence.

NOTATION. Let $h: G \times G \to H$ and $\lambda: G \to H$ be maps. Then $h_\lambda: G \times G \to H$ is a map defined by

$$h_\lambda(x, y) = \lambda(x) \cdot {}^x\lambda(y) \cdot h(x, y) \cdot \lambda(xy)^{-1}$$

where ${}^x\lambda(y) = \Phi(\phi(x))(\lambda(y))$ for certain ϕ which should be clear from the context.

Throughout this section, we shall work with *a fixed 2-cocycle* (ϕ, h) with coefficient in the crossed module (H, ρ, Π, Φ) . This induces a cocycle $(\bar{\rho}, \bar{h}) = (\tau \circ \phi, j \circ h) \in Z^2(G; \bar{H})$.

We introduce a concept of relative cohomology. Let A be a subgroup of Z which is Π-invariant; that is, A is invariant under the group $\Phi(\Pi) \subset \operatorname{Aut}(H)$. Two 1-cocycles $\lambda, \eta \in Z^1_{(\phi,h)}(G; H)$ are *A-cohomologous* if there exists $a \in A$ for which $\eta(x) = a \cdot \lambda(x) \cdot {}^x a^{-1}$ for all $x \in G$. We denote the set of A-cohomology classes of $Z^1_{(\phi,h)}(G; H)$ by $H^1_{(\phi,h)}(G; H, A)$.

The map ϕ induces a homomorphism $\operatorname{restr} \circ \Phi \circ \phi: G \to \Pi \to \operatorname{Aut}(H) \to \operatorname{Aut}(A)$, since A is Π-invariant. There is an action of $H^1_{ab}(G; A)$ on the set $H^1_{(\phi,h)}(G; H, A)$. Let $\nu: G \to A$ and $\lambda: G \to H$ be elements of $Z^1_{ab}(G; A)$ and $Z^1_{(\phi,h)}(G; H)$, respectively. The action of ν on λ is defined by the multiplication $\nu \cdot \lambda: G \to H$. The two identities $\nu(x) \cdot {}^x\nu(y) \cdot \nu(xy)^{-1} = 1$ and $\lambda(x) \cdot {}^x\lambda(y) \cdot h(x, y) \cdot \lambda(xy)^{-1} = 1$ show that $\nu \cdot \lambda$ satisfies $(\nu \cdot \lambda)(x) \cdot {}^x(\nu \cdot \lambda)(y) \cdot h(x, y) \cdot (\nu \cdot \lambda)(xy)^{-1} = 1$ so that $\nu \cdot \lambda \in Z^1_{(\phi,h)}(G; H)$. Suppose $\nu, \nu' \in Z^1_{ab}(G; A)$ and $\lambda, \lambda' \in Z^1_{(\phi,h)}(G; H)$ such that $\nu \sim \nu'$, and $\lambda \sim \lambda'$ relative to A. Let $a, b \in A$ such that $\nu'(x) = a \cdot \nu(x) \cdot {}^x a^{-1}$ and $\lambda'(x) = b \cdot \lambda(x) \cdot {}^x b^{-1}$ for every $x \in G$. Then $(\nu' \cdot \lambda')(x) = ab \cdot (\nu \cdot \lambda)(x) \cdot {}^x(ab)^{-1}$, which shows that $\nu' \cdot \lambda'$ is A-cohomologous to $\nu \cdot \lambda$. Thus we have shown that $(\nu, \lambda) \mapsto \nu \cdot \lambda$ defines an *action* of $H^1_{ab}(G; A)$ on $H^1_{(\phi,h)}(G; H, A)$. See [**D**, **LR** or **I**] for actions similar to this.

Now we state the main result of this section, the existence of the "long exact sequence" of cohomology sets of G with coefficient in crossed modules.

THEOREM 2.1. *Let*

$$\begin{array}{ccccccccc} & & \Pi & \xrightarrow{\text{id}} & \Pi & \xrightarrow{\tau} & \bar{\Pi} & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & K & \xrightarrow{i} & H & \xrightarrow{j} & \bar{H} & \to & 1 \end{array}$$

be a short exact sequence of crossed modules. Suppose that $Z^2(G;H)$ is nonempty, and choose $(\phi, h) \in Z^2(G;H)$. Then there is a long exact sequence (in the sense of (1) and (2) below)

$$H^1_{ab}(G;Z) \overset{\text{ev}}{\to} H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z}) \overset{\delta}{\to} H^2(G;K) \overset{i}{\to} H^2(G;H).$$

($\bar{\phi} = \tau \circ \phi$, $\bar{h} = j \circ h$, *and* $\bar{Z} = j(Z)$, $Z = \text{Kernel}(\rho)$), *where the first map is induced by the action of $H^1_{ab}(G;Z)$ on $H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z})$, through $H^1_{ab}(G;Z) \to H^1_{ab}(G;\bar{Z})$.*

(1) *Let $\langle\bar{\lambda}\rangle, \langle\bar{\eta}\rangle \in H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z})$. Then $\delta\langle\bar{\lambda}\rangle = \delta\langle\bar{\eta}\rangle$ if and only if $\langle\bar{\eta}\rangle = \nu \cdot \langle\bar{\lambda}\rangle$ for some $\langle\nu\rangle \in H^1_{ab}(G;Z)$. In particular, if $H^1_{ab}(G;Z) = 0$, then δ is injective.*

(2) *$\langle\psi, k\rangle \in H^2(G;K)$ is in the image of δ if and only if $i(\langle\psi, k\rangle) = \langle\phi, h\rangle$. In particular, if $i^{-1}(\langle\phi, h\rangle)$ is nonempty, then $H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z})$ is also nonempty.*

PROOF. Our first task is to define the connecting map δ. Let $\bar{\lambda} \in Z^1_{(\bar{\phi},\bar{h})}(G;\bar{H})$. Take a lift $\lambda: G \to H$ so that $\bar{\lambda} = j \circ \lambda$. Consider $h_\lambda: G \times G \to H$ defined by

$$h_\lambda(x,y) = \lambda(x) \cdot {}^x\lambda(y) \cdot h(x,y) \cdot \lambda(xy)^{-1}.$$

Since $\bar{\lambda}$ is a 1-cocycle with respect to $(\bar{\phi}, \bar{h})$, $\bar{\lambda}(x) \cdot {}^x\bar{\lambda}(y) \cdot \bar{h}(x,y) \cdot \bar{\lambda}(xy)^{-1} = 1$. This implies that $h_\lambda(x,y) \in K$. It is easy to see that $((\rho \circ \lambda) \cdot \phi, h_\lambda) \in Z^2(G;K)$. We shall define $\delta\langle\bar{\lambda}\rangle$ by $\langle(\rho \circ \lambda) \cdot \phi, h_\lambda\rangle$.

In order to show δ is well defined, let λ' be another lift of $\bar{\lambda}$. Then $\lambda' = \xi \cdot \lambda$ for some $\xi: G \to K$. Then one can show that

$$(\rho \circ \lambda') \cdot \phi = (\rho \circ \xi) \cdot ((\rho \circ \lambda) \cdot \phi), \quad h_{\lambda'} = (h_\lambda)_\xi.$$

The second equality means that

$$h_{\lambda'}(x,y) = \xi(x) \cdot {}^{\rho(\lambda(x))\cdot\phi(x)}\xi(y) \cdot h_\lambda(x,y) \cdot \xi(xy)^{-1}.$$

These show that $((\rho \circ \lambda') \cdot \phi, h_{\lambda'})$ is cohomologous to $((\rho \circ \lambda) \cdot \phi, h_\lambda)$ in $Z^2(G;K)$.

We still have to show that if $\bar{\eta} \in Z^1_{(\bar{\phi},\bar{h})}(G;\bar{H})$ is $\bar{Z}$-cohomologous to $\bar{\lambda}$, then $((\rho \circ \eta) \cdot \phi, h_\eta)$ is cohomologous to $((\rho \circ \lambda) \cdot \phi, h_\lambda)$ for some (and hence, for any) lifts η, λ of $\bar{\eta}, \bar{\lambda}$, by the above paragraph. Let $\bar{z} \in \bar{Z}$ be such that $\bar{\eta}(x) = \bar{z} \cdot \bar{\lambda}(x) \cdot {}^x\bar{z}^{-1}$ for all $x \in G$. Choose $z \in Z$ with $j(z) = \bar{z}$, and a lift $\lambda: G \to H$ of $\bar{\lambda}$. Put

$$\eta(x) = z \cdot \lambda(x) \cdot {}^x z^{-1}.$$

Then $\eta: G \to H$ is a lift of $\bar{\eta}$. (Recall that we are free to choose any lift of $\bar{\eta}$). Clearly, $(\rho \circ \eta) \cdot \phi = (\rho \circ \lambda) \cdot \phi$. Furthermore, it is not hard to verify that $h_\eta = h_\lambda$. Thus we have proved that $((\rho \circ \eta) \cdot \phi, h_\eta)$ is cohomologous to $((\rho \circ \lambda) \cdot \phi, h_\lambda)$. This proves the well-definedness of the map δ.

Now we shall verify (1). Let $\langle\bar{\lambda}\rangle, \langle\bar{\eta}\rangle \in H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z})$. Suppose $\langle\bar{\eta}\rangle = \nu \cdot \langle\bar{\lambda}\rangle$ for some $\langle\nu\rangle \in H^1_{ab}(G;Z)$. Choose a lift λ. Then $\eta = \nu \cdot \lambda$ is a lift of $\bar{\eta}$. We have to show that $\delta\bar{\eta} = ((\rho \circ \eta) \cdot \phi, h_\eta)$ is cohomologous to $\delta\bar{\lambda} = ((\rho \circ \lambda) \cdot \phi, h_\lambda)$. Both equalities $(\rho \circ \eta) \cdot \phi = (\rho \circ \lambda) \cdot \phi$ and $h_\eta = h_\lambda$ can be verified easily using the fact ν is an abelian cocycle with values in the center of H. Therefore, $\delta\langle\bar{\eta}\rangle = \delta\langle\bar{\lambda}\rangle$. Note that $H^1_{ab}(G, Z)$ cannot be replaced by $H^1_{ab}(G, \bar{Z})$ in this argument.

Conversely, let $\bar{\lambda}, \bar{\eta} \in Z^1_{(\bar{\phi},\bar{h})}(G; \bar{H})$ such that $\delta\langle\bar{\eta}\rangle = \delta\langle\bar{\lambda}\rangle$; that is, $((\rho \circ \eta) \cdot \phi, h_\eta)$ is cohomologous to $((\rho \circ \lambda) \cdot \phi, h_\lambda)$ for some lifts λ and η. Then there is a map $\xi: G \to K$ satisfying

$$(\rho \circ \eta) \cdot \phi = (\rho \circ \xi) \cdot (\rho \circ \lambda) \cdot \phi, \qquad h_\eta = (h_\lambda)_\xi.$$

From the first equality, we have $\rho \circ \eta = \rho \circ (\xi \cdot \lambda)$. Thus, there exists a map $\nu: G \to Z$ so that $\eta = \nu \cdot \xi \cdot \lambda$. Moreover, from the second equality, we have

$$h_\eta(x, y) = \nu(x) \cdot {}^x\nu(y) \cdot \nu(xy)^{-1} \cdot h_\eta(x, y)$$

for all $x, y \in G$. Thus, $\nu(x) \cdot {}^x\nu(y) \cdot \nu(xy)^{-1} = 1$ so that $\nu: G \to Z$ is an abelian cocycle. We have proved that if $\delta\langle\bar{\eta}\rangle = \delta\langle\bar{\lambda}\rangle$, then $\eta = \nu \cdot \xi \cdot \lambda$, and hence $\bar{\eta} = \bar{\nu} \cdot \bar{\lambda}$, where $\bar{\nu} = j(\nu) = j \circ \nu$ with $\nu \in Z^1_{ab}(G; Z)$.

To prove (2), suppose $\delta\langle\bar{\lambda}\rangle = \langle\psi, k\rangle$. For a choice of lift $\lambda: G \to H$, we have $(\psi, k) \sim ((\rho \circ \lambda) \cdot \phi, h_\lambda)$ in K. This means (ψ, k) and $((\rho \circ \lambda) \cdot \phi, h_\lambda)$ are cohomologous as cocycles with coefficient in H. Thus, $i\langle\psi, k\rangle = \langle\phi, h\rangle$. Conversely, let $\langle\psi, k\rangle \in H^2(G; K)$ with $i\langle\psi, k\rangle = \langle\phi, h\rangle$. Then there exists $\lambda: G \to H$ so that $\psi = (\rho \circ \lambda) \cdot \phi$ and $k = h_\lambda$. Now it is easy to see that $\lambda \in Z^1_{(\bar{\phi},\bar{h})}(G, \bar{H})$. This completes the proof of the theorem.

3. Second cohomology vs. group extension. As in the ordinary case, the second cohomology with coefficient in a crossed module is related to group extensions. Let (H, ρ, Π, Φ) be a crossed module and G a group. A Π-*crossed extension* (E, σ) of H by G is a group extension $1 \to H \xrightarrow{i} E \xrightarrow{j} G \to 1$ together with a homomorphism $\sigma: E \to \Pi$ satisfying $\sigma \circ i = \rho$ and $\Phi \circ \sigma = \mu$, where $\mu: E \to \mathrm{Aut}(H)$ is the natural homomorphism defined by conjugations. Two such Π-crossed extensions (E, σ) and (E', σ') are said to be Π-*equivalent* if there is an isomorphism $\theta: E' \to E$ inducing the identity maps on H and G, and such that $\sigma \circ \theta = \theta'$.

A 2-cocycle $(\phi, h) \in Z^2(G; H)$ gives rise to an extension E by setting $E = H \times G$ with group operation $(a, x) \cdot (b, y) = (a \cdot {}^x b \cdot h(x, y), xy)$. Moreover, one can define a homomorphism $\sigma: E \to \Pi$ by $\sigma(a, x) = \rho(a) \cdot \phi(x)$. Then the pair (E, σ) is a Π-crossed extension of H by G. Conversely, a Π-crossed extension (E, σ) gives rise to a 2-cocycle $(\phi, h) \in Z^2(G; H)$. This can be seen as follows. Choose any map $s: G \to E$ so that $j \circ s = \mathrm{id}$, and define ϕ and h by $\phi = \sigma \circ s$ and $h(x, y) = s(x) \cdot s(y) \cdot s(xy)^{-1}$. This (ϕ, h) is a desired 2-cocycle. In fact, the following is well known. See [**D**, **LR**].

PROPOSITION 3.1. *For a crossed module* (H, ρ, Π, Φ) *and a group* G, *there is a one-one correspondence between* $H^2(G; H)$ *and the set of* Π-*equivalence classes of* Π-*crossed extensions.*

Not every group extension of a group H by G represents an element of $H^2(G; H)$. For example, consider a crossed module $(H, \mu, \mathrm{Inn}(H), \mathrm{incl.})$. If a group extension of H by G represents an element of $H^2(G; H)$, then the induced abstract kernel, $G \to \mathrm{Out}(H)$, is trivial. In other words, a group extension of H by G cannot be a Π-crossed extension, unless the induced abstract kernel is trivial. We have a necessary condition.

LEMMA 3.2. *Let (H, ρ, Π, Φ) be a crossed module. Suppose there exists a homomorphism $r: \mathrm{Aut}(H) \to \Pi$ such that $\Phi \circ r = \mathrm{id}$ and $r \circ \mu = \rho$. Then every group extension of H by G is a Π-crossed extension, and hence represents an element of $H^2(G; H)$. Furthermore, two extensions are equivalent (in the ordinary sense) if and only if they are Π-equivalent.*

PROOF. Let $1 \to H \to E \to G \to 1$ be any group extension. Define $\sigma: E \to \Pi$ by $\sigma = r \circ \mu$, where $\mu: E \to \mathrm{Aut}(H)$ is the natural homomorphism (=conjugation). Then clearly $\sigma \circ i = \rho$ and $\Phi \circ \sigma = \mu$. The rest of the statement is easy to verify.

LEMMA 3.3. *Let $(\phi, h) \in Z^2(G; H)$. Then ϕ induces a homomorphism $\phi_Z: G \to \mathrm{Aut}(Z)$ naturally. If $(\psi, k) \in Z^2(G; H)$ is cohomologous to (ϕ, h), then $\psi_Z = \phi_Z$.*

PROOF. Clearly, $Z = \mathrm{Ker}(\rho)$ is Φ-invariant. Therefore, one can define ϕ_Z as the restriction of $\Phi(\phi(x))$ to Z. To show that ϕ_Z is a homomorphism, let $x, y \in G$ and $z \in Z$. Recall that $\phi(x)\phi(y) = \rho(h(x,y))\phi(xy)$ from the definition of a cocycle. Now since $\Phi \circ \rho = \mu$, we find that $\phi_Z(x) \circ \phi_Z(y) = \phi_Z(xy)$. Thus $\phi_Z: G \to \mathrm{Aut}(Z)$ is a homomorphism, and hence Z is a G-module via ϕ_Z.

Suppose (ψ, k) is cohomologous to (ϕ, h). Then there exists a map $\lambda: G \to H$ for which

$$\psi = (\rho \circ \lambda) \cdot \phi, \quad k = h_\lambda.$$

Then $\psi_Z(x)(z) = \Phi(\psi(x))(z) = \Phi[\rho(\lambda(x)) \cdot \phi(x)](z) = \lambda(x) \cdot \Phi(\phi(x))(z) \cdot \lambda(x)^{-1} = \Phi(\phi(x))(z) = \phi_Z(z)$ since $\Phi(\phi(x))(z) \in Z$. This completes the proof of the lemma.

One of the biggest disadvantages of the nonabelian cohomology is that it is extremely hard to calculate it. We attempt to "convert" a nonabelian cohomology to an abelian one.

$H^2_\phi(G; H)$ is the subset of $H^2(G; H)$ consisting of elements of the form $\langle \phi, - \rangle$. The following is not surprising. Compare this with the situation of $\mathrm{Ext}(G, H)$ in [**M**].

PROPOSITION 3.4. *Let (H, ρ, Π, Φ) be a crossed module. Suppose $H^2_\phi(G; H)$ is nonempty. Then there is a simply transitive action of $H^2_{ab}(G; Z)$ on $H^2_\phi(G; H)$, where $Z = \mathrm{Ker}(\rho)$ has the G-module structure via ϕ_Z. Consequently, $H^2_\phi(G; H) \approx H^2_{ab}(G; Z)$.*

PROOF. Let $\langle f \rangle \in H^2_{ab}(G; Z)$ and $\langle \phi, h \rangle \in H^2_\phi(G; H)$. Define $\langle f \rangle \cdot \langle \phi, h \rangle$ by $\langle \phi, f \cdot h \rangle$. Write $\Phi(\phi(x))(a)$ by ${}^x a$, for $x \in G$ and $a \in H$. Since $f: G \times G \to Z$ is a cocycle, ${}^x f(y,z) \cdot f(x, yz) = f(x,y) \cdot f(xy, z)$. We claim that $\langle \phi, f \cdot h \rangle \in Z^2(G; H)$. Let $k = f \cdot h$. Then

$$\phi(x)\phi(y) = \rho(h(x,y)) \cdot \phi(xy) = \rho(f(x,y) \cdot h(x,y)) \cdot \phi(xy) = \rho(k(x,y)) \cdot \phi(xy)$$

where the second equality holds since $f(x,y) \in Z = \mathrm{Ker}(\rho)$. The second condition for the nonabelian 2-cocycle, ${}^x k(y,z) \cdot k(x, yz) = k(x,y) \cdot k(xy, z)$, follows readily from the facts that f is an abelian cocycle, h is a cocycle and that f has image in the center of H. Thus we have verified that $\langle \phi, f \cdot h \rangle \in H^2_\phi(G; H)$.

In order to show that the map $\langle f \rangle \mapsto \langle \phi, f \cdot h \rangle$ is well defined, let $f \sim f'$ in $Z^2_{ab}(G; Z)$. Then there is a map $\lambda: G \to Z$ such that $f' = f_\lambda$. We would like to show that $(\phi, f \cdot h) \sim (\phi, f' \cdot h)$. Since λ has value in $\mathrm{Ker}(\rho)$, $\phi = (\rho \circ \lambda) \cdot \phi$. Moreover, $f' \cdot h = f_\lambda \cdot h = (f \cdot h)_\lambda$. These two equalities show that $(\phi, f \cdot h) \sim (\phi, f' \cdot h)$.

We proceed to show that the correspondence $\langle f\rangle \mapsto \langle \phi, f\cdot h\rangle$ is one-to-one. Let $f, f' \in Z^2_{ab}(G;Z)$ for which $(\phi, f\cdot h) \sim (\phi, f'\cdot h)$. Then there exists a map $\lambda: G \to H$ with $\phi = (\rho\circ\lambda)\cdot\phi$ and $f'(x,y)\cdot h(x,y) = \lambda(x)\cdot {}^x\lambda(y)\cdot f(x,y)\cdot h(x,y)\cdot\lambda(xy)^{-1}$. From the first equality, we see that $\lambda(x)\in Z$ for all $x\in G$ so that $\lambda: G\to Z$. From this fact and the second identity, it is easy to see that $f\sim f'$ in $Z^2_{ab}(G;Z)$.

Let $(\phi,k)\in Z^2_\phi(G;H)$. Put $f = k\cdot h^{-1}: G\times G\to H$. Since $(\phi,h),(\phi,k)\in Z^2_\phi(G;H)$, we have

$$\rho(h(x,y))\cdot\phi(xy) = \phi(x)\cdot\phi(y) = \rho(k(x,y))\cdot\phi(xy)$$

so that $\rho h(x,y) = \rho k(x,y)$. This implies $f(x,y)\in Z$ for all $x,y\in G$. Now it is not hard to see that $f: G\times G\to Z$ is an abelian 2-cocycle. Thus the action is transitive.

A 2-cocycle $(\phi,h)\in Z^2(G;H)$ induces an abstract kernel

$$\phi_{\text{out}}: G \xrightarrow{\phi} \Pi \xrightarrow{\Phi} \text{Aut}(H) \longrightarrow \text{Out}(H)$$

where the last map is the canonical homomorphism. Let $q(\Pi) = \Pi/\text{Ker}(\Phi)$; $q:\Pi\to q(\Pi)$, the quotient map; $q(\Phi): q(\Pi)\to\text{Aut}(H)$, the homomorphism induced from Φ so that $q(\Phi)\circ q = \Phi$. Then $(H, q\circ\rho, q(\Pi), q(\Phi))$ is a crossed module with $q(\Phi)$ injective. Clearly $q = (\text{id}, q): (H,\rho,\Pi,\Phi)\to(H, q\circ\rho, q(\Pi), q(\Phi))$ is a morphism of crossed modules.

LEMMA 3.5. *Let* $(\phi,h),(\psi,k)\in Z^2(G;(H,\rho,\Pi,\Phi))$. *Then* $\psi_{\text{out}} = \phi_{\text{out}}$ *if and only if* $q\circ(\phi,h)\sim q\circ(\psi,k)$ *in* $Z^2(G;(H,q\circ\rho,q(\Pi),q(\Phi)))$.

PROOF. Suppose $(q\circ\phi,h)\sim(q\circ\psi,k)$. Then there is $\lambda: G\to H$ for which $q\circ\psi = ((q\circ\rho)\circ\lambda)\cdot(q\circ\phi) = (q\circ(\rho\circ\lambda)\cdot\phi)$. This implies that $\Phi\circ\psi = \Phi\circ((\rho\circ\lambda)\cdot\phi) = \Phi\circ\phi$ so that $\psi_{\text{out}} = \phi_{\text{out}}$. The converse is true by reversing the order of the above argument.

Suppose an abstract kernel $G\to\text{Out}(H)$ is given. In general, many disjoint factors of $H^2(G;H) = \bigcup H^2_\psi(G;H)$ may be represented by group extensions of H by G yielding the same abstract kernel. In other words, $H^2_\phi(G;H)\cap H^2_\psi(G;H)$ may be empty, and yet it is possible that $\psi_{\text{out}} = \phi_{\text{out}}$. However, if Φ is injective, this does not happen.

COROLLARY 3.6. *If* $(\phi,h)\sim(\psi,k)$, *then* $\psi_{\text{out}} = \phi_{\text{out}}$. *The converse is also true if* Φ *is injective.*

COROLLARY 3.7. *In Theorem* (2.1), *assume that* Φ *is injective. If*

$$\bigcup\{H^2_\psi(G;K): \psi_{\text{out}} = \phi_{\text{out}}\}$$

is nonempty, then there exists a long exact sequence

$$H^1_{ab}(G;Z) \xrightarrow{\text{ev}} H^1_{(\bar\phi,\bar h)}(G;\bar H,\bar Z) \xrightarrow{\delta} \bigcup\{H^2_\psi(G;K): \psi_{\text{out}} = \phi_{\text{out}}\}$$
$$\xrightarrow{i} H^2_\phi(G;H) \cong H^2_{ab}(G;Z).$$

PROOF. In the exact sequence in (2.1), restrict the last cohomology set to the subset $H^2_\phi(G;H)$, which is in one-one correspondence with $H^2_{ab}(G;Z)$ by (3.4). It remains to understand the inverse image of this set under the map i. Consider $(\psi,k)\in Z^2(G;K)$, as a 2-cocycle in $Z^2(G;H)$; that is, $i(\psi,k) = (\psi,k)$. By (3.6), this is cohomologous to (ϕ,h) if and only if $\psi_{\text{out}} = \phi_{\text{out}}$ as maps from G into $\text{Out}(H)$, since Φ is injective. This finishes the proof.

4. Application to group pairs. Throughout this section, let K be a normal subgroup of a group H. The subgroup of $\mathrm{Aut}(H)$ leaving K invariant is denoted by $\mathrm{Aut}(H,K)$. Let $\Pi = \mathrm{Aut}(H,K)$, $\rho\colon K \to \mathrm{Aut}(H,K) = \Pi$ the conjugation map, and $\Phi\colon \Pi \to \mathrm{Aut}(K)$ be the usual restriction homomorphism. Then (K, ρ, Π, Φ) is a crossed module. Of course, $(H, \mu, \Pi, \mathrm{incl})$ is a crossed module with center $Z = Z(H)$, where $\mathrm{incl}\colon \Pi = \mathrm{Aut}(H,K) \hookrightarrow \mathrm{Aut}(H)$ is the inclusion homomorphism. Now let

$$\bar{\Pi} = \mathrm{Aut}(H,K)/\mathrm{Inn}(K), \quad \bar{H} = H/K$$

and

$$\bar{\mu}\colon \bar{H} \to \bar{\Pi}, \quad \bar{\Phi}\colon \bar{\Pi} \to \mathrm{Aut}(\bar{H})$$

be the homomorphisms induced by $\mu\colon H \to \Pi$ and $\Phi = \mathrm{incl}\colon \Pi \to \mathrm{Aut}(H)$. Then $(\bar{H}, \bar{\mu}, \bar{\Pi}, \bar{\Phi})$ is a crossed module. The following is easy.

LEMMA 4.1. *Let K, H, Φ, and $\bar{\Phi}$ be as above. Then*

$$\begin{array}{ccccccccc} & & \Pi & \xrightarrow{\mathrm{id}} & \Pi & \xrightarrow{\tau} & \bar{\Pi} & & \\ & & {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle\bar{\mu}} & & \\ 1 & \longrightarrow & K & \xrightarrow{i} & H & \xrightarrow{j} & \bar{H} & \longrightarrow & 1 \end{array}$$

is a short exact sequence of crossed modules.

For the rest of this section, we *fix an extension* $\mathcal{E}$ so that $1 \to H \to \mathcal{E} \to G \to 1$ is exact and that K is *normal* in $\mathcal{E}$. Then $\mathcal{E}$ is a Π-crossed extension so that it represents an element $(\phi, h) \in Z^2(G; H)$. This, in turn, induces $(\bar{\phi}, \bar{h}) \in Z^2(G; \bar{H})$. Consider the set of all commutative diagrams with exact rows

$$(*) \qquad \begin{array}{ccccccccc} 1 & \to & K & \longrightarrow & E & \longrightarrow & G & \to & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow & & \\ 1 & \to & H & \longrightarrow & \mathcal{E} & \longrightarrow & G & \to & 1 \end{array}$$

Two such diagrams (E, θ) and (E', θ') are *equivalent up to conjugation* if there is an isomorphism $\omega\colon E \to E'$ and an element c of H such that

(i) $\omega\,|_K = \mathrm{id}$ on K and ω induces the identity on G, and

(ii) $\mu(c) \circ \theta = \theta' \circ \omega$.

This is an equivalence relation on the set of all pairs (E, θ). We denote the equivalence class of (E, θ) by $[E, \theta]$.

THEOREM 4.2. *There is a one-to-one correspondence between $H^1_{(\bar{\phi},\bar{h})}(G; \bar{H}, \bar{Z})$ and the set of equivalence classes $[E, \theta]$.*

PROOF. Recall the definition of the connecting map $\delta\colon H^1_{(\bar{\phi},\bar{h})}(G; \bar{H}, \bar{Z}) \to H^2(G; K)$. Let $\bar{\lambda} \in Z^1_{(\bar{\phi},\bar{h})}(G; \bar{H})$. Take a lift $\lambda\colon G \to H$. Then $((\rho \circ \lambda) \cdot \phi, h_\lambda) \in Z^2(G; K)$. Let $1 \to K \to E \to G \to 1$ be an extension associated with $((\rho \circ \lambda) \cdot \phi, h_\lambda)$. More precisely, E is the set $K \times G$ with the group law

$$(a, x) \cdot (b, y) = (a \cdot {}^{\rho(\lambda(x))\phi(x)}b \cdot h_\lambda(x, y), xy).$$

Note that $\mathcal{E}$ also can be described as the set $H \times G$ with the group law

$$(a, x) \cdot (b, y) = (a \cdot {}^{\phi(x)}b \cdot h(x, y), xy)$$

since $\mathcal{E}$ represents $(\phi, h) \in Z^2(G; H)$. We define $\theta: E \to \mathcal{E}$ by $\theta(a, x) = (a \cdot \lambda(x), x)$. Calculations show that θ is a homomorphism inducing the original inclusion map on K and on G.

If we chose a different lift λ' of $\bar{\lambda}$, then $\lambda' = \xi \cdot \lambda$ for some map $\xi: G \to K$, and we will have a new cocycle $((\rho \circ \lambda') \cdot \phi, h_{\lambda'})$. Let E' be the extension of K by G corresponding to $((\rho \circ \lambda') \cdot \phi, h_{\lambda'})$. Then

$$\omega: (a, x) \mapsto (a \cdot \xi(x)^{-1}, x)$$

defines an isomorphism of E onto E'. Furthermore, it is easy to see that

$$(\theta' \circ \omega)(a, x) = \theta'(a \cdot \xi(x)^{-1}, x) = (a \cdot \lambda(x), x) = \theta(a, x)$$

so that $\theta' \circ \omega = \theta$.

Now let $\bar{\lambda}, \bar{\eta} \in Z^1_{(\bar{\phi}, \bar{h})}(G; \bar{H})$ with $\langle \bar{\lambda} \rangle = \langle \bar{\eta} \rangle$ in $H^1_{(\bar{\phi}, \bar{h})}(G; \bar{H}, \bar{Z})$. There is $\bar{z} \in \bar{Z}$ for which $\bar{\eta}(x) = \bar{z} \cdot \bar{\lambda}(x) \cdot {}^x\bar{z}^{-1}$ for all $x \in G$. By the previous paragraph, one is free to choose any lift of $\bar{\lambda}$ and of $\bar{\eta}$. Therefore, we may assume the lifts λ, η of $\bar{\lambda}$, $\bar{\eta}$ satisfy

$$\eta(x) = z \cdot \lambda(x)^x z^{-1}$$

for some $z \in Z$. Let (E, θ), (E', θ') be the extensions together with embeddings corresponding to λ, η, respectively. Then they are realizations of the cocycles $((\rho \circ \lambda) \cdot \phi, h_\lambda)$, $((\rho \circ \eta) \cdot \phi, h_\eta)$, respectively. Clearly, $(\rho \circ \eta) \cdot \phi = (\rho \circ \lambda) \cdot \phi$ and $h_\eta = h_\lambda$. This means that $\bar{\lambda}$, $\bar{\eta}$ define the same extension but different embeddings. We show that (E, θ) is equivalent to (E', θ'). Consider the diagram

$$\begin{array}{ccc} E & \stackrel{\omega=\mathrm{id}}{\longrightarrow} & E' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \mathcal{E} & \stackrel{\mu(z,1)}{\longrightarrow} & \mathcal{E} \end{array}$$

where $\theta(a, x) = (a \circ \lambda(x), x)$, $\theta'(a, x) = (a \circ \eta(x), x)$. It is easy to see that

$$\theta' \circ \omega = \mu(z, 1) \circ \theta.$$

Therefore, the sequence above is commutative. We conclude that (E, θ) is equivalent to (E', θ'). Thus we have shown that $H^1_{(\bar{\phi}, \bar{h})}(G; \bar{H}, \bar{Z}) \to \{[E, \theta]\}$ is well defined.

In order to show that the correspondence is surjective, we start from a diagram (E, θ) in $(*)$, where the bottom exact sequence is a realization of the fixed cocycle (ϕ, h).

Choose a section $s: G \to E$ (not a homomorphism in general). Define $\psi: G \to \Pi = \mathrm{Aut}(H, K)$ by the composite $G \to E \to \mathcal{E} \to \mathrm{Aut}(H, K)$, where the last homomorphism is given by the conjugation. (Recall that K is normal in $\mathcal{E}$.) We also define $k: G \times G \to K$ by $k(x, y) = s(x) \cdot s(y) \cdot s(xy)^{-1}$. Then $(\psi, k) \in Z^2(G; K)$. Now one can view E as a realization of (ψ, k). More precisely, $E = K \times s(G)$ with the group law

$$(a, x)(b, y) = (a \cdot \Phi(\psi(x))(b) \cdot k(x, y), xy).$$

Since $\theta\,|_K: K \to \mathcal{E}$ is the original inclusion map of K into H, and θ induces the identity on G, θ is of the form

$$\theta(a, x) = (a \cdot \lambda(x), x)$$

for some $\lambda: G \to H$. It is not hard to see that $\psi = (\rho \circ \lambda) \cdot \phi$ since $\Phi: \Pi = \mathrm{Aut}(H,K) \to \mathrm{Aut}(H)$ is *injective.* Also it is clear that $k = h_\lambda$. In other words, (E,θ) is a realization of the cocycle $((\rho\circ\lambda)\cdot\phi, h_\lambda)$. Let $\bar{\lambda} = j \circ \lambda: G \to \bar{H}$. Then $\bar{\lambda} \in Z^1_{(\bar{\phi},\bar{h})}(G;\bar{H})$. This shows that the pair (E,θ) came from $\langle\bar{\lambda}\rangle \in H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z})$.

Now it remains only to show that the correspondence $\langle\bar{\lambda}\rangle \mapsto [E,\theta]$ is injective. Let $\bar{\lambda} \mapsto (E,\theta)$, $\bar{\lambda}' \mapsto (E',\theta')$, and suppose that $[E,\theta] = [E',\theta']$. Then there is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\omega} & E' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \mathcal{E} & \xrightarrow{\mu(z,1)} & \mathcal{E} \end{array}$$

for some $z \in Z$. Recall that $(E,\theta), (E',\theta')$ have the corresponding cocycles $((\rho\circ\lambda)\cdot\phi, h_\lambda)$, $((\rho\circ\lambda')\cdot\phi, h_{\lambda'})$, respectively. Since $\omega \mid K$ is the identity, and ω induces the identity on G, ω is of the form $\omega(a,x) = (a\cdot\xi(x)^{-1}, x)$ for some $\xi: G \to K$. Then

$$(a\cdot\xi(x)^{-1}\cdot\lambda'(x), x) = (\theta'\circ\omega)(a,x) = (\mu(z,1)\circ\theta)(a,x) = (z\cdot a\cdot\lambda(x)\cdot {}^x z^{-1}, x).$$

So

$$\lambda'(x) = \xi(x)\cdot z\cdot\lambda(x)\cdot {}^x z^{-1}$$

which implies that $\bar{\lambda}'(x) = \bar{z}\cdot\bar{\lambda}(x)\cdot {}^x\bar{z}^{-1}$ so that $\langle\bar{\lambda}\rangle = \langle\bar{\lambda}'\rangle$. See [L] for a similar construction in the case of abelian groups.

Let K be a normal subgroup of H. We shall say that a pair (H,K) has the *automorphism extension property* (= AEP) if there is a homomorphism $r: \mathrm{Aut}(K) \to \mathrm{Aut}(H,K)$ so that $\mathrm{restr} \circ r = \mathrm{id}$ and $r \circ \mu = \mu'$, where $\mu: K \to \mathrm{Aut}(K)$ and $\mu': K \to \mathrm{Aut}(H,K)$ are both given by conjugations. There is an induced homomorphism $\bar{r}: \mathrm{Out}(K) \to \mathrm{Out}(H,K) = \mathrm{Aut}(H,K)/\mathrm{Inn}(K)$. Therefore, any homomorphism $G \to \mathrm{Out}(K)$ (called an abstract kernel) induces a homomorphism $G \to \mathrm{Out}(K) \to \mathrm{Out}(H,K) \subset \mathrm{Out}(H)$. Also from (3.3), $Z(H)$ has the Q-module structure that is given by $\mu_{Z(H)}$.

The following is the main result of this paper. It is important for the geometric construction in the last section.

THEOREM 4.3. *Let K be a normal subgroup of H with AEP, and let $1 \to H \longrightarrow \mathcal{E} \longrightarrow G \to 1$ be an exact sequence of groups. If $H^2_{ab}(G; Z(H)) = 0$, then for any group extension $1 \to K \to E \to G \to 1$ whose abstract kernel inducing the abstract kernel for $\mathcal{E}$, there exists an injective homomorphism $\theta: E \to \mathcal{E}$ making the diagram $(*)$ commutative. If, furthermore, $H^1_{ab}(G; Z(H)) = 0$, then such a homomorphism θ is unique up to conjugation by elements of H.*

PROOF. We shall apply (3.2), (3.7) and (4.2). Since K is normal in $\mathcal{E}$, $\mathcal{E}$ is a Π-crossed extension. Let $(\phi, h) \in Z^2(G;H)$ be the cocycle represented by $\mathcal{E}$. Since $\Phi: \mathrm{Aut}(H,K) \to \mathrm{Aut}(H)$ is injective, one can apply (3.7) to have the long exact sequence

$$H^1_{ab}(G;Z) \xrightarrow{\text{ev}} H^1_{(\bar{\phi},\bar{h})}(G;\bar{H},\bar{Z}) \xrightarrow{\delta} \bigcup\{H^2_\psi(G;K): \psi_{\text{out}} = \phi_{\text{out}}\}$$
$$\xrightarrow{i} H^2_\phi(G;H) \cong H^2_{ab}(G;Z).$$

First, we show that (ψ, h) satisfies $\psi_{\text{out}} = \phi_{\text{out}}$. The AEP ensures the existence of $r: \mathrm{Aut}(H) \to \Pi$ in (3.2). Therefore, the extension E is a Π-crossed extension. Let

$(\psi, k) \in Z^2(G; K)$ be a cocycle corresponding to E. Since the abstract kernel for E induces the abstract kernel for $\mathcal{E}$, $\psi_{\text{out}} = \phi_{\text{out}}: G \to \text{Out}(H)$.

By the assumption, $H^2_\phi(G; H)$ contains only one element $\langle \phi, h \rangle$. Therefore, $i\langle \psi, k \rangle = \langle \phi, h \rangle$. By the exactness, there exists an element $\langle \bar{\lambda} \rangle \in H^1_{(\bar{\phi}, \bar{h})}(G; \bar{H}, \bar{Z})$. Now by (4.2), there is an injective homomorphism $\theta: E \to \mathcal{E}$ making the diagram $(*)$ commutative.

If $H^1_{ab}(G; Z) = 0$, then δ is injective. This means that there is only one cohomology class that hits $\langle \psi, k \rangle$. By the identification $\langle \psi, k \rangle$ with the equivalence class $[E, \theta]$, $\theta: E \to \mathcal{E}$ is the only homomorphism making $(*)$ commutative, up to conjugation. This completes the proof of theorem.

5. Seifert fiber spaces. Let L be a Lie group, W a connected, simply connected space. Suppose there is a group Q acting properly discontinuously on W. Let E be an extension of L by Q. We are interested in finding an action of E on the product space $L \times W$ in a natural way.

The group of all continuous maps of W into L is denoted by $\mathcal{M} = \mathcal{M}(W, L)$. We shall use the following group law on $\mathcal{M}$:

$$(\lambda_1 * \lambda_2)(w) = \lambda_2(w) \cdot \lambda_1(x) \; (= (\lambda_2 \cdot \lambda_1)(w))$$

for $\lambda_1, \lambda_2 \in \mathcal{M}$, $w \in W$, where the multiplication on the right takes place in the Lie group L. The group $\text{Aut}(L) \times \mathcal{H}(W)$ acts on $\mathcal{M}$, where $\mathcal{H}(W)$ denotes the group of self-homeomorphisms of W, by

$$(g, h) \cdot \lambda = g \circ \lambda \circ h^{-1}.$$

Form the semidirect product $\mathcal{M}(W, L) \circ (\text{Aut}(L) \times \mathcal{H}(W))$. The group law is

$$(\lambda_1, g_1, h_1)(\lambda_2, g_2, h_2) = ((g_1 \circ \lambda_2 \circ h_1^{-1}) \cdot \lambda_1, g_1 g_2, h_1 h_2).$$

It is easily checked that

$$(\lambda, g, h)(x, w) = (g(x) \cdot \lambda(hw), hw)$$

for $(x, w) \in L \times W$, defines an action of $\mathcal{M} \circ (\text{Aut}(L) \times \mathcal{H}(W))$ on $L \times W$. Thus it becomes a subgroup of $\mathcal{H}(L \times W)$. It is known that $\mathcal{M} \circ (\text{Aut}(L) \times \mathcal{H}(W))$ is the normalizer of $l(L)$, the group of left translations of $L \times W$, in the group of the self-homeomorphisms of the space $L \times W$. We denote this semidirect product by $\mathcal{H}^F(L \times W)$. For more details about this group, the reader is referred to [**KLR**].

Consider a subgroup $\mathcal{S}$ of $\mathcal{M}$ which contains all the constant maps $r(L)$. Then $\mathcal{S} \circ \text{Inn}(L)$ contains the left translations $l(L)$ via $(a, \mu(a))$.

LEMMA 5.1. *The pair $(\mathcal{S} \circ \text{Inn}(L), l(L))$ has the AEP.*

PROOF. Let $f: L \to L$ be an automorphism. Define $\tilde{f}: \mathcal{S} \circ \text{Inn}(L) \to \mathcal{S} \circ \text{Inn}(L)$ by

$$\tilde{f}(\lambda, \mu(x)) = (f(x) \cdot x^{-1} \cdot \lambda, \mu(f(x))).$$

It is easy to see that $\tilde{f}$ is an automorphism of $\mathcal{S} \circ \text{Inn}(L)$. Furthermore, for any $x \in L$, $\tilde{f}(x, \mu(x)) = (f(x), \mu(f(x))) = f(x, \mu(x))$, which shows that $\tilde{f}$ is an extension of f.

Define $r: \text{Aut}(L) \to \text{Aut}(\mathcal{S} \circ \text{Inn}(L), L)$ by $r(f) = \tilde{f}$. Then one can check that r is a homomorphism which is a right inverse of the restriction homomorphism

$\mathrm{Aut}(\mathcal{S} \circ \mathrm{Inn}(L), L) \to \mathrm{Aut}(L)$. Suppose $f = \mu(a) \in \mathrm{Inn}(L)$. Then clearly $\tilde{f} = \mu(a, \mu(a))$. Therefore, $r \circ \mu = \mu'$. See the definition of AEP for the notation.

By (5.1), we can now apply the theory in §4. With $K = l(L) = L; H = \mathcal{S} \circ \mathrm{Inn}(L)$; and $\Pi = \mathrm{Aut}(\mathcal{S} \circ \mathrm{Inn}(L), L)$, we have a short exact sequence of crossed modules

$$1 \to L \to \mathcal{S} \circ \mathrm{Inn}(L) \to \mathcal{S} \circ \mathrm{Inn}(L)/L \cong \mathcal{S}/Z(L) \to 1.$$

Furthermore, the middle crossed module $(\mathcal{S} \circ \mathrm{Inn}(L), \rho, \mathrm{Aut}(\mathcal{S} \circ \mathrm{Inn}(L), L), \Phi)$ has injective Φ.

Let $\rho: Q \to \mathcal{H}(W)$ be an action. Suppose that $1 \to L \to E \to Q \to 1$ is exact. The abstract kernel $\varphi: Q \to \mathrm{Out}(L)$ induces a homomorphism $\hat{\varphi}: Q \to \mathrm{Aut}(Z(L))$.

NOTATION. $Z = \mathcal{M}(W, Z(L)) \cap \mathcal{S}$, $I = \mathrm{Inn}(L)$.

Clearly, Z is the center of $\mathcal{S} \circ \mathrm{Inn}(L)$. Now Z becomes a Q-module via

$$\alpha \cdot \lambda = \hat{\varphi}(\alpha) \circ \lambda \circ \rho(\alpha)^{-1}$$

for $\alpha \in Q$, $\lambda \in Z$.

THEOREM 5.2. *Let $\mathcal{S}$ be a subgroup of $\mathcal{M}(W, L)$ containing all the constant maps $r(L)$ and invariant under the action of* $\mathrm{Aut}(L) \times \rho(Q)$. *Let $1 \to L \to E \to Q \to 1$ be exact. Equip $Z = \mathcal{M}(W, Z(L)) \cap \mathcal{S}$ with the Q-module structure $\hat{\varphi}$. Suppose that the abstract kernel of E induces the abstract kernel of $\mathcal{E}$. If $H^2(Q; Z) = 0$, then there is a commutative diagram with exact rows*

$$(**) \qquad \begin{array}{ccccc} 1 \to L & \longrightarrow & E & \longrightarrow & Q \to 1 \\ \downarrow & & \downarrow & & \downarrow \\ 1 \to \mathcal{S} \circ I & \to & \mathcal{S} \circ (A \times \mathcal{H}) & \to & \mathcal{O} \times \mathcal{H} \to 1 \end{array}$$

where $I = \mathrm{Inn}(L), A = \mathrm{Aut}(L), \mathcal{H} = \mathcal{H}(W)$. *If, furthermore, $H^1(Q; Z) = 0$, then such a homomorphism θ is unique up to conjugation by elements of* $\mathcal{S} \circ \mathrm{Inn}(L)$.

PROOF. By (5.1), the pair $(\mathcal{S} \circ \mathrm{Inn}(L), L)$ has the AEP. We assume that the homomorphism $\varphi \times \rho$ of Q into $\mathrm{Out}(L) \times \mathcal{H}(W)$ is injective. The general case follows easily by a slight modification and hence the details are omitted. Let $1 \to \mathcal{S} \circ \mathrm{Inn}(L) \to \mathcal{E} \to Q \to 1$ be the exact sequence obtained by the image of Q in $\mathrm{Out}(L) \times \mathcal{H}(W)$ from the bottom sequence of the theorem. Of course, we are using the homomorphism which consists of the abstract kernel of the top exact sequence and the action of Q on the space W. Now with $L = K, \mathcal{S} \circ I = H$ and $Q = G$, all the hypotheses in Theorem 4.3 are satisfied. It is interesting to see that the homomorphism $\mathrm{Out}(L) \to \mathrm{Out}(\mathcal{S} \circ I)$ is injective.

EXAMPLES. Let Q act on a connected, simply connected space W properly discontinuously in the sense of [**CR**]. It is known that

$$H^i(Q; \mathcal{M}(W, \mathbf{R}^k)) = 0,$$
$$H^i(Q; C(W, \mathbf{R}^k)) = 0$$

for $i > 0$, where $C(W, \mathbf{R}^k)$ is the group of all smooth maps in the case when W has a smooth structure.

Let L be any connected Lie group with a connected center. Then $Z(L) \cong \mathbf{R}^k$ for some $k \geq 0$. Therefore, $H^i(Q; \mathcal{M}(W, Z(L))) = 0$ for $i = 1, 2$. This implies then that any extension of L by Q can be mapped into $\mathcal{M}(W, L) \circ (\mathrm{Aut}(L) \times \mathcal{H}(W))$ naturally.

For example, suppose L is a connected, simply connected nilpotent Lie group. Then $Z(L)$ is connected. Suppose $1 \to \Gamma \to \pi \to Q \to 1$ is exact, where Γ is a finitely generated torsion-free nilpotent group. Then take the connected, simply connected nilpotent Lie group containing Γ as a uniform lattice (i.e., L is a Malcev completion of Γ). Since any automorphism of Γ can be uniquely extended to an automorphism of L, there is an exact sequence $1 \to L \to E \to Q \to 1$ with commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \Gamma & \longrightarrow & \pi & \longrightarrow & Q & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & L & \longrightarrow & E & \longrightarrow & Q & \to & 1 \end{array}$$

By our theory, there exists a commutative diagram (**) with $\mathcal{M} = \mathcal{S}$. Consequently, π acts on $L \times W$ as fiber-preserving homeomorphisms. When W/Q is compact and the π-action on $L \times W$ is free, then $\pi \backslash L \times W$ is a manifold with a Seifert fiber structure whose typical fiber is the nilmanifold $\Gamma \backslash L$, and singular fibers are infranilmanifolds covered by the typical fiber.

By the fact that $H^1(Q; \mathcal{M}(W, Z(L))) = 0$, any two such Seifert fiber spaces are homeomorphic through a fiber preserving homeomorphism. This has been discussed in [**KLR**] in detail.

REMARK. The requirement for abstract kernels in (5.2) is not innocuous. For a more general case, the readers are referred to a forthcoming paper [**LR2**].

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